Spectral Geometry of Operator Polynomials and Applications to QFT

Dmitri V. Fursaev *

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
141980 Dubna, Moscow Region, Russia

February 1, 2008

Abstract

A class of non-linear eigenvalue problems defined in the form of operator polynomials is investigated. The problems are related to wave equations which appear in a relativistic quantum field theory. Spectral asymptotics for this class are found explicitly. The properties of operator polynomials are analyzed for scalar, spinor and gauge fields. It is also shown how to use these results in finite temperature theories.

Key words: polynomial operator pencils, spectral asymptotics, quantum effects in external fields

*fursaev@thsun1.jinr.ru

1 Introduction

Consider a class of eigenvalue problems which have the following form:

$$P(\omega)\phi_\omega \equiv (P_n + \omega P_{n-1} + ... + \omega^{n-1}P_1 + \omega^n P_0)\phi_\omega = 0 \quad , \tag{1.1}$$

where ω is a complex spectral parameter and ϕ_ω is an eigenfunction. P_k $(k = 0, ..., n)$ are partial differential operators of k-th order acting on a Hilbert space $\mathcal{H}$. $P(\omega)$ represents an example of what is called the operator polynomial or polynomial operator pencil [1]. In general P_k and P_l do not commute if $k \neq l$ and (1.1) is essentially non-linear problem with respect to the eigenvalue ω.

Problems like (1.1) appear in different applications (oscillations in a viscous fluid, Schrödinger equation with energy dependent potential and etc), see [1] for the introduction and references. The focus of the present work is on applications related to quantum field theory.

Consider equation of motion for a free field $\phi(x, x^0)$ (x and x^0 denote spatial and time coordinates, respectively). It can be always written as

$$P(i\partial_0)\phi(x, x^0) \equiv (P_2 + P_1(i\partial_0) + P_0(i\partial_0)^2)\phi(x, x^0) = 0 \quad , \tag{1.2}$$

where $\partial_0 = \partial/\partial_0$. Operators P_k depend only on spatial derivatives and may have additional indexes related to a spin-tensor structure. Field ϕ propagates in an external classical background. If the background is stationary the variables can be separated by choosing $\phi(x, x^0) = e^{-i\omega x^0}\phi_\omega(x)$. From (1.2) one gets

$$P(\omega)\phi_\omega(x) = (P_2 + \omega P_1 + \omega^2 P_0)\phi_\omega(x) = 0 \quad . \tag{1.3}$$

This is a quadratic operator polynomial, a particular case of problem (1.1).

Eigenvalues ω determine the spectrum of energies of single-particle excitations which is used to compute a number of important physical quantities, like the vacuum (Casimir) energy $E = \pm\frac{1}{2}\sum|\omega|$, free energy and etc.

Consider for instance a charged Dirac field ψ in an external static electromagnetic field with the potential $A_\mu = A_\mu(x)$. The equation is

$$[\gamma^\mu D_\mu + m]\psi = 0 \quad , \tag{1.4}$$

where $D_\mu = \nabla_\mu - ieA_\mu$ and e is the charge. To get for the single-particle spectrum problem like (1.3) one has to act by the operator $[\gamma^\mu D_\mu - m]$ on the left hand side (l.h.s.) of (1.4) and make substitution $\psi(x, x^0) = e^{-i\omega x^0}\psi_\omega(x)$. This yields

$$[D_k D^k - m^2 + e^2 A_0^2 - \frac{e}{2}(\mathcal{F} - 2i\gamma^0\gamma^k\partial_k A_0) + \omega\ 2eA_0 + \omega^2]\psi_\omega(x) = 0 \quad , \tag{1.5}$$

where $\mathcal{F} = (\partial_j A_k - \partial_k A_j)\gamma^j\gamma^k$ and $k, j = 1, 2, 3$. When the time component A_0 of the potential is a function of spatial coordinates (1.5) cannot be reduced to a standard eigenvalue problem because operator $P_1 = 2eA_0$ does not commute with P_2.

Sec. 2 of this contribution contains a summary of results obtained in [2, 3]. The aim is to show that the notion of spectral geometry can be extended to a class of quadratic operator polynomials which appear in problems of quantum field theory. Applications to finite temperature theories are discussed in Sec. 3 where, as an example, the obtained results are used to describe the Debye screening in hot plasma. Finally, some important features of operator polynomials for scalar, spinor and gauge fields are analyzed in Sec. 4.

2 Spectral Geometry of Quadratic Polynomials

To describe the results we rewrite (1.3) in the form

$$\left[\omega^2 - L(\omega)\right]\phi_\omega(x) = 0 \quad , \tag{2.1}$$

$$L(\omega) = L_2 + \omega L_1 + \omega^2 L_0 \quad , \tag{2.2}$$

where $L(\omega) = \omega^2 - P(\omega)$ and L_k are partial differential operators of the k-th order. Suppose for a moment that ω is a complex parameter and consider a one-parameter family of operators $L(\omega)$. Let $\Lambda_k(\omega)$ be eigenvalues of $L(\omega)$ for a given ω,

$$L(\omega)\phi^{(\omega)}_{\Lambda_k} = \Lambda_k(\omega)\phi^{(\omega)}_{\Lambda_k}. \tag{2.3}$$

If the spectrum $\Lambda_k(\omega)$ is known for any ω, the spectrum of problem (2.1) is determined by roots of equation

$$\chi(\omega,\Lambda_k) = 0, \tag{2.4}$$

$$\chi(\omega,\Lambda_k) = \omega^2 - \Lambda_k(\omega). \tag{2.5}$$

Denote these roots by $\omega_{k,i}$ and define the following function on the spectrum

$$\chi'(\omega_{k,i}) = \partial_\omega\chi(\omega_{k,i},\Lambda_k) \quad \text{where} \quad \omega^2_{k,i} = \Lambda_k(\omega_{k,i}). \tag{2.6}$$

It is assumed that at first the derivative in $\chi'(\omega_{k,i})$ over ω is taken for the fixed branch of eigenvalues $\Lambda_k(\omega)$ and after that the result is considered on the corresponding root $\omega_{k,i}$. In what follows we write ω instead of $\omega_{k,i}$.

We make three assumptions: i) $L(\omega)$ is a Laplace type operator of the form:

$$L(\omega) = -(\nabla_k + iA_k + i\omega a_k)(\nabla^k + iA^k + i\omega a^k) + \omega B + V \tag{2.7}$$

which acts on sections to vector bundles over a d-dimensional compact Riemannian manifold $\mathcal{M}_d$; ii) the spectrum of $L(0)$ is strictly positive; iii) the function $\chi'(\omega)$, see (2.6), is positive (negative) on positive (negative) eigenvalues ω.

First two conditions are technical (see details in [3]). The origin of condition (iii) is explained in Sec. 4. At this point we note that in the limit when the polynomial is trivial, $L(\omega) = L_2$, this condition is satisfied because $\chi'(\omega) = 2\omega$.

To describe the spectrum of (2.1) at large $|\omega|$ we introduce the pseudo-trace

$$K(t) = \frac{1}{2} \sum_{\omega} e^{-t\omega^2}, \quad t > 0, \tag{2.8}$$

where summation goes over the real eigenvalues ω. The asymptotics of $K(t)$ at small t is related to the distribution of large $|\omega|$. Given conditions (i)–(iii) the following asymptotic series takes place at small t [3]

$$K(t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[a_n t^n + b_n t^{n+1/2} \right]. \tag{2.9}$$

a_n and b_n can be computed by using expansion formula for the heat kernel

$$K_\omega(t) = \mathrm{Tr}\ e^{-tL(\omega)} \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[a_n(\omega) t^n + b_n(\omega) t^{n+1/2} \right] \tag{2.10}$$

which is well-known [4]. For operators defined in (2.7),

$$a_n(\omega) = \sum_{m=0}^{n} a_{m,n} \omega^m, \quad b_n(\omega) = \sum_{m=0}^{n} b_{m,n} \omega^m \tag{2.11}$$

and it can be shown that [3]

$$a_n = \sum_{m=n}^{2n} (-1)^{n-m} \frac{\Gamma\left(-\frac{d}{2} + m\right)}{\Gamma\left(-\frac{d}{2} + n\right)} a_{2(m-n),m}, \tag{2.12}$$

$$b_n = \sum_{m=n}^{2n} (-1)^{n-m} \frac{\Gamma\left(-\frac{d-1}{2} + m\right)}{\Gamma\left(-\frac{d-1}{2} + n\right)} b_{2(m-n),m}. \tag{2.13}$$

Coefficient a_0 is proportional to the volume of $\mathcal{M}^d$, other a_n, b_n are integrals of local functionals of background fields and b_n are non-trivial when $\mathcal{M}^d$ has a boundary. The notion of spectral geometry does extend to a class of quadratic operator polynomials.

The proof of (2.9), (2.12), (2.13) is based on relation [3]

$$K(t) = \frac{1}{4\pi} \int_0^\infty d\omega \int_C dz\ e^{-\omega^2(t-iz)} \left(2\omega + \frac{1}{iz}\partial_\omega\right) (K_\omega(iz) + K_{-\omega}(iz)), \tag{2.14}$$

where a contour C goes in the complex plane just below the real axis $\mathrm{Im}\ z = 0$. Thus $K(t)$ is determined by part of $K_\omega(t)$ which is symmetric with respect to the transformation ω to $-\omega$. This property is a direct consequence of condition (iii) which we imposed on the spectrum of the operator polynomial. This also explains why coefficients a_n and b_n in (2.12), (2.13) are determined by the symmetric part of coefficients $a_n(\omega)$ and $b_n(\omega)$.

3 Applications to Finite-Temperature Theories

Spectral asymptotics of quadratic operator polynomials can be used in a number of physical applications. One of them is studying the behaviour of the free energy of non-interacting quanta at high temperatures. The free energy for Fermi particles at temperature $T = \beta^{-1}$ is defined as

$$F(\beta) = -\beta^{-1} \sum_k \ln \left(1 + e^{-\beta E_k}\right) . \tag{3.1}$$

Summation goes over single-particle energies $E_k = |\omega_k|$ which are determined by the real eigenvalues ω_k of the corresponding wave equations.

It is convenient to start with the case when the single-particle spectrum is determined by the standard linear eigenvalue problem, i.e., when operator $L(\omega) = L_2$ and it does not depend on ω. This case is well studied. It is known that in three dimensions $d = 3$ the free energy in the high temperature limit has the following asymptotic expansion [5]

$$F(\beta, A) \sim -\frac{7\pi^2}{720} T^4 a_0 - \frac{1}{48} T^2 a_1 - \frac{1}{16\pi^2} \ln(T/\varrho) a_2 + O(T^{-2}) \tag{3.2}$$

(we neglect boundary terms for simplicity). Here a_n are the coefficients of the asymptotic expansion of the trace Tr e^{-tL_2}, ϱ is a dimensional parameter.

In the high temperature limit $F(\beta)$ is determined by behaviour of the single-particle spectrum at large $|\omega|$. For polynomial (2.1) this behaviour is described by asymptotic formula (2.9) which has exactly the same form as expansion of Tr e^{-tL_2}. Therefore, for systems whose spectrum is determined by (2.1) formula (3.2) can be used if a_n are calculated with the help of (2.12) (analogously for boundary terms if they are present).

Let us use this method to derive the free energy of electron-positron plasma in an external static electromagnetic field $A_\mu(x)$. (Finite-temperature theories in gravitational backgrounds are discussed in [2]). Coefficients a_n are determined by (2.12) for problem (1.5). The corresponding operator is

$$L(\omega) = -\mathcal{D}_k \mathcal{D}^k + m^2 - e^2 A_0^2 + \frac{e}{2}(\mathcal{F} - 2i\gamma^0\gamma^k \partial_k A_0) - 2eA_0\omega \ . \tag{3.3}$$

By using standard results [4] one easily finds coefficients of $L(\omega)$ and obtains

$$a_0 = a_{0,0} = 4 \int_{\mathcal{M}^3} d^3x \ \ , \tag{3.4}$$

$$a_1 = a_{0,1} + \frac{1}{2} a_{2,2} = 4 \int_{\mathcal{M}^3} d^3x (-m^2 + 2e^2 A_0^2) \ \ . \tag{3.5}$$

$$a_2 = a_{0,2} - \frac{1}{2} a_{2,3} + \frac{3}{4} a_{4,4} = \frac{2e^2}{3} \int_{\mathcal{M}^3} d^3x F_{\mu\nu} F^{\mu\nu} \ , \tag{3.6}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Consider the effective thermodynamic potential of the system at high temperature by neglecting the zero-point energy

$$\Omega(\beta, A) = -\mathrm{L(A)} + \mathrm{F}(\beta, \mathrm{A}) = \frac{1}{4} \int_{\mathcal{M}^3} \mathrm{d^3x F_{\mu\nu} F^{\mu\nu}} + \mathrm{F}(\beta, \mathrm{A}) \ \ , \tag{3.7}$$

where L(A) is the classical Lagrangian of the background potential A_μ. Eqs. (3.2), (3.4)–(3.6) yield

$$\Omega(\beta, A) = -\int_{\mathcal{M}^3} d^3x \left(-\frac{c(T)}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2(T) A_0^2\right) \ , \tag{3.8}$$

$$c(T) = 1 - \frac{e^2}{24\pi^2} \ln(T/\rho) \ \ , \ \ m^2(T) = \frac{1}{3} e^2 T^2 \ . \tag{3.9}$$

These formulas reproduce the result which is well-known in plasma physics. The last term in (3.8) is responsible for screening effects in hot plasma and $m(T)$ is the correct expression for the Debye mass.

Note that conventional derivation of (3.8) is based on computation of one-loop diagrams at finite temperature in the limit of vanishing gauge fields. Our calculation is done for arbitrary static gauge potential.

Coefficient a_2 in (3.6) has a Lorentz invariant structure. For this reason the logarithmic term in expansion (3.2) results in a finite renormalization of the background potential (see the first term in r.h.s. of (3.8)). The Lorentz invariant structure of a_2 is not a mere coincidence. It reflects a more general property established in [3]: coefficient a_2 of the pseudo-trace expansion coincides with the coefficient A_2 of the asymptotic expansion of the heat kernel of the covariant four-dimensional operator $P(i\partial_0) = -\partial_0^2 - L(i\partial_0)$.

4 Operator Polynomials for Different Field Theories

Spectral asymptotics given in Sec. 2 are obtained by using certain conditions. In particular, the structure of coefficients a_n and b_n in (2.12), (2.13) is determined by condition (iii). Let us analyze the origin of this condition for models of non-interacting fields.

Consider a pair f_1, f_2 of solutions to field Eq. (1.2) (or (1.4), for Dirac case) and introduce the relativistic product, a Hermitian bilinear form $\langle f_1, f_2\rangle = \langle f_2, f_1\rangle^*$ defined on a space-like hypersurface Σ in a hyperbolic spacetime $\mathcal{M}$ as

$$\langle f_1, f_2\rangle = \int_\Sigma d\Sigma^\mu \ j_\mu(f_1, f_2) \ . \tag{4.1}$$

The "current" $j_\mu(f_1, f_2)$ is divergence free, $\nabla^\mu j_\mu = 0$, and (4.1) does not depend on Σ.

If the background is stationary one can introduce a conserved canonical energy $H[f]$ for a given solution f. It can be shown that for free scalar, spinor and gauge fields the energy can be written in the form

$$H[f] = \frac{i}{2}\langle f, \partial_0 f\rangle + c.c. \ , \tag{4.2}$$

where "*c.c.*" is a term obtained from the first term in the r.h.s. of (4.2) by complex conjugation. It follows from (4.2) that for a solution f_ω with certain frequency ω ($i\partial_0 f_\omega = \omega f_\omega$)

$$H[f_\omega] = \omega\langle f_\omega, f_\omega\rangle \ . \tag{4.3}$$

In general the system may have excitations with complex frequencies ω. It is easy to see that they have vanishing relativistic norm $\langle f_\omega, f_\omega \rangle$ and, hence, the vanishing energy. Such modes along with zero-frequency modes are not quantized and should be considered separately from real-frequency solutions. This is the reason why pseudo-trace (2.8) is defined for real ω.

4.1 Scalar Fields

Consider integer spin fields. To have a well-defined theory we require that canonical energy is non-negative,

$$H[f_\omega] \geq 0 \quad , \tag{4.4}$$

for any ω. If there are single-particle excitations with negative energy H the Fock space in the corresponding quantized theory contains states with negative energies unbounded from below.

Condition (iii) of Sec. 2 for scalar fields ϕ follows from (4.4). Indeed it is possible to show [3] that in this case

$$\langle \phi_\omega, \phi_\omega \rangle = \chi'(\omega)(\phi_\omega, \phi_\omega) \ , \tag{4.5}$$

where $\chi'(\omega)$ was introduced in (2.6) and $(\phi_\omega, \phi_\omega) = \int dV |\phi_\omega|^2$ is a positive-definite norm in a Hilbert space of single-particle wave functions. (4.4) together with relations (4.3), (4.5) implies that $\chi'(\omega) = \varepsilon(\omega)|\chi'(\omega)|$ where $\varepsilon(\omega)$ is the sign function.

4.2 Gauge Fields

We discuss now theory of non-Abelian gauge fields with the gauge group $SU(N)$. Let B_μ be a solution to Yang-Mills equations, $\mathcal{D}^\mu F^{(B)}_{\mu\nu} = 0$, where $F^{(B)}_{\mu\nu}$ is the strength tensor of B_μ (we work in the adjoint representation). In the linear order, perturbations A_μ of the Yang-Mills field on the background B_μ obey the equation

$$\mathcal{D}^\nu \mathcal{D}_\nu A_\mu - \mathcal{D}_\mu \mathcal{D}^\nu A_\nu + 2iF^{(B)}_{\mu\nu} A^\nu = 0 \ , \tag{4.6}$$

where $\mathcal{D}_\mu = \partial_\mu + iB_\mu$. It is convenient to work in the background gauge $\mathcal{D}_\mu A^\mu = 0$ where equation transforms to

$$\mathcal{D}^\nu \mathcal{D}_\nu A_\mu + 2iF^{(B)}_{\mu\nu} A^\nu = 0 \quad . \tag{4.7}$$

Fixing this gauge leaves a freedom in the gauge transformations $\delta A_\mu = \mathcal{D}_\mu \lambda$ where λ is a solution to

$$\mathcal{D}_\mu \mathcal{D}^\mu \lambda = 0 \quad . \tag{4.8}$$

Suppose now that B_μ is a stationary field. Then Eqs. (4.6)–(4.8) result in operator polynomials of the form (1.3). Let ω be the physical spectrum of (4.6), $\omega_{(1)}$ the spectrum related to vector Eq. (4.7) and $\omega_{(0)}$ the spectrum of (4.8). It is clear that ω's represent

a subset among eigenvalues $\omega_{(1)}$. The pseudo-trace for the physical spectrum can be represented as

$$K(t)=\sum_{\omega>0}e^{-t\omega^2}=K_{(1)}(t)-2K_{(0)}(t)\equiv\sum_{\omega_{(1)}>0}e^{-t\omega_{(1)}^2}-2\sum_{\omega_{(0)}>0}e^{-t\omega_{(0)}^2}\quad . \tag{4.9}$$

(For $SU(N)$ group Eqs. (4.6)–(4.8) are invariant with respect to the charge conjugation and their spectrum is symmetric with respect to change ω to $-\omega$.) The double subtraction of the sum over $\omega_{(0)}$ in (4.9) has the following explanation. One subtraction eliminates solutions of (4.7) which do not respect the gauge conditions, $\mathcal{D}_\mu A^\mu=\phi\neq 0$. It is easy to see that such modes have frequencies $\omega_{(0)}$ because ϕ obeys scalar Eq. (4.8). Additional subtraction of $K_{(0)}(t)$ in (4.9) eliminates pure gauge solutions $A_\mu=\mathcal{D}_\mu\lambda$. The last term in r.h.s of (4.9) can be interpreted as contribution of ghosts.

There is an analog of formula (4.5) for transverse solutions ($\mathcal{D}_\mu A^\mu=0$) to (4.6). The relativistic norm for the solutions with energy ω can be written as

$$\langle A_\omega,A_\omega\rangle=\chi'(\omega)(A_\omega,A_\omega)\quad , \tag{4.10}$$

where $(A_\omega,A_\omega)=\int dV(A^+_\omega)^\mu(A_\omega)_\mu$ and quantity $\chi'(\omega)$ is defined for the polynomial corresponding to vector problem (4.7). The residual gauge freedom can be used to impose condition $A_0=0$. For such solutions the norm (A_ω,A_ω) becomes positive-definite.

Suppose that the canonical energy of perturbations A_μ on the background B_μ is non-negative, $H_{YM}[A]\geq 0$. Require also the same property for canonical energy of the scalar field described by (4.8), $H[\lambda]\geq 0$. Then one can show that operator polynomials associated with vector, (4.7), and scalar, (4.8), problems obey condition (iii) of Sec. 2. The proof is based on relations (4.5), (4.10).

4.3 Dirac Field

As is known, the classical energy for the Dirac field is not positive-definite. Thus, the proof of condition (iii) in this case should be different. Consider the Dirac field in Minkowsky space-time in the presence of a static gauge potential A_μ. The relativistic norm of a single-particle wave function is $\langle\psi,\psi\rangle=\int dV\ \psi^+\psi$. If the field has a mass, $m\neq 0$, one can prove the relation [3]

$$\langle\psi_\omega,\psi_\omega\rangle=\frac{\chi'(\omega_1)}{2m}(\psi_\omega,\psi_\omega)\ , \tag{4.11}$$

where $(\psi_\omega,\psi_\omega)=\int dV\ \psi^+_\omega i\gamma_0\psi_\omega$ and γ_0 is anti-Hermitian. It follows from (4.11) and positivity of the norm $\langle\psi,\psi\rangle$ that $\chi'(\omega)$ does not vanish for massive fields. Now, if $A_\mu=0$, one has $\chi'(\omega)=2\omega$ and condition (iii) is satisfied. The validity of (iii) for $A_\mu\neq 0$ can be proved by continuity. Because $\chi'(\omega)\neq 0$ quantity $\chi'(\omega)$ cannot change the sign when A_μ is switched on from zero to some value.

Acknowledgments

The author is grateful to the Organizing Committee of the 6th Workshop on "Quantum Field Theory under the Influence of External Conditions" for the financial support.

References

[1] A.S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Translations of Mathematical Monographs, vol. 71, American Mathematical Society, Providence, R.I, 1988.

[2] D.V. Fursaev, Nucl. Phys. **B596**, 365 (2001), Erratum-ibid. **B664**, 403 (2003), D.V. Fursaev, *Nucl. Phys. B (Proc. Suppl.)* **104**, 33 (2002).

[3] D.V. Fursaev, *Class. Quantum Grav.* **19**, 3635 (2002), Erratum-ibid. **20**, 565 (2003).

[4] P.B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, CRC Press, Boca Raton, 1995.

[5] J.S. Dowker and J.P. Schofield, Nucl. Phys. **B327**, 267 (1989).